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Analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations on a finite domain

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ABSTRACT

Generalized fractional partial differential equations have now found wide application for describing important physical phenomena, such as subdiffusive and superdiffusive processes. However, studies of generalized multi-term time and space fractional partial differential equations are still under development. In this paper, the multi-term time–space Caputo–Riesz fractional advection diffusion equations (MT-TSCR-FADE) with Dirichlet nonhomogeneous boundary conditions are considered. The multi-term time-fractional derivatives are defined in the Caputo sense, whose orders belong to the intervals $[0,1]$, $[1,2]$ and $[0,2]$, respectively. These are called respectively the multi-term time-fractional diffusion terms, the multi-term time-fractional wave terms and the multi-term time-fractional mixed diffusion-wave terms. The space fractional derivatives are defined as Riesz fractional derivatives. Analytical solutions of three types of the MT-TSCR-FADE are derived with Dirichlet boundary conditions. By using Luchko's Theorem (Acta Math. Vietnam., 1999), we proposed some new techniques, such as a spectral representation of the fractional Laplacian operator and the equivalent relationship between fractional Laplacian operator and Riesz fractional derivative, that enabled the derivation of the analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations.

1. Introduction

In recent years, fractional partial differential equations have attracted great attention [16,26]. The fractional advection–diffusion equation is presented as a useful approach for the description of transport dynamics in complex systems that are governed by anomalous diffusion and non-exponential relaxation patterns [23]. The time, space and time–space fractional advection–dispersion equations have been used to describe important physical phenomena that arise in amorphous, colloid, glassy and porous media, in fractals and percolation clusters, comb structures, dielectrics and semiconductors, biological systems, polymers, random and disordered media, geophysical and geological processes (see [3,8,19,23,24,41]) and in groundwater hydrology to model the transport of passive tracers carried by fluid flow in a porous medium [44].

It is noteworthy to mention recent important progress in numerically simulating fractional advection–diffusion equations. Meerschaert and Tadjeran [21] presented numerical methods to solve the one-dimensional fractional advection–diffusion equation with a Riemann–Liouville fractional derivative on a finite domain. Roop [34] investigated the numerical approximation of the variational solution to the fractional advection–diffusion equation on boundary domains in $\mathbb{R}^2$. Liu et al. [11]

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transformed the space fractional advection–diffusion equation into a system of ordinary differential equations (method of lines) that was then solved using backward differentiation formulas. Liu et al. [12] also considered a space–time fractional advection–diffusion equation with a Caputo time-fractional derivative and Riemann–Liouville space fractional derivatives. Shen et al. [36] discussed the fundamental solution and discrete random walk model for the time–space Riesz fractional advection–dispersion equation. Shen et al. [37] also investigated the fundamental solution and numerical solution of the Riesz fractional advection–dispersion equation. Shen et al. [38] proposed numerical approximations and solution techniques for the space–time Riesz–Caputo fractional advection–diffusion equation.

Analytical solutions of fractional advection–diffusion/fractional advection–diffusion-wave equations are of fundamental importance in describing and understanding dispersion phenomena, since all the parameters are expressed in a mathematically closed form and therefore the influence of individual parameters on pollutant concentration can be easily examined [25]. Also, the analytical solutions make it easy to obtain asymptotic behaviours of the solutions, that are usually difficult to obtain through numerical calculations.

The fundamental solution for the fractional diffusion-wave equation in one space dimension was obtained by Mainardi [18]. Agrawal [1] analyzed the fractional diffusion-wave equation in a half-line and a bounded domain $[0, L]$. Gorenflo and Mainardi [7] studied the time-fractional, spatially one-dimensional diffusion-wave equation on the spatial half-line with zero initial conditions. They considered the Dirichlet and Neumann boundary conditions and proved that the Dirichlet–Neumann map is given by a time-fractional differential operator whose order is one-half the order of the time-fractional derivative. Mainardi and Paradisi [20] employed the time-fractional diffusion-wave equation to study the propagation of stress waves in viscoelastic media relevant to acoustics and seismology. Several initial and boundary-value problems for the time-fractional diffusion-wave equation were investigated by Povstenko [27–31]. He also considered the generalized Cattaneo-type equations with time- and space-fractional derivatives [32]. Qi and Jiang also presented the exact solution of the fractional Cattaneo equation [33]. Golbabai and Sayevand [5] investigated fractional advection dispersion equation by means of the homotopy perturbation method with consideration of a promising scheme to calculate nonlinear terms. Zhou and Jiao [45] discussed the nonlocal Cauchy problem for the fractional evolution equations in an arbitrary Banach space and obtained various criteria on the existence and uniqueness of mild solutions. By applying fractional calculus to model the behavior of cells and tissues, Magin et al. [17] unraveled the inherent complexity of individual molecules and membranes in a way that leads to an improved understanding of the overall biological function and behavior of living systems. Shakhmurov and Shahmurova [35] studied the boundary value problems for linear and nonlinear degenerate elliptic differential operator equations of a second order. By using these results the existence, uniqueness and the maximal regularity of boundary value problems for nonlinear degenerate parabolic differential operator equations are established. Wang et al. [42] developed a fast characteristic finite difference method for the efficient solution of space-fractional transient advection–diffusion equations in one space dimension. Meerschaert et al. [22] derived explicit strong solutions and stochastic analogues for distributed-order time-fractional diffusion equations on bounded domains, with Dirichlet boundary conditions.

Daftardar-Gejji and Bhalekar [4] considered a multi-term fractional diffusion-wave equation along with the homogeneous/nonhomogeneous boundary conditions and this equation has been solved using the method of separation of variables. Based on orthogonal polynomials of the Laguerre type, Stojanovic [39] found solutions for the diffusion-wave problem in one dimension with n-term time-fractional derivatives whose orders belong to the intervals $(0, 1)$, $(1, 2)$ and $(0, 2)$, respectively. Equations in this paper can be solved in one-dimension by the method of the approximation of the tempered convolution. This method transfers the diffusion-wave problem into the corresponding infinite system of linear algebraic equations through the coefficients, which are uniquely solvable under some relations between the coefficients with index zero. Luchko [15] considered the initial–boundary-value problems for the generalized time-fractional diffusion equation over an open bounded domain. Based on an appropriate maximum principle, Luchko [15] derived some *a priori* estimates for the solution and then its uniqueness is established using the Fourier method of the separation of variables. The time-dependent components of the solution are given in terms of the multinomial Mittag–Leffler function.

Time domain wave-equations for lossy media obeying a frequency power-law were discussed by Kelly et al. [10]. Mathematically, the power-law frequency dependence of the attenuation coefficient cannot be modeled with standard dissipative partial differential equations with integer-order derivatives. Kelly et al. [10] considered the Szabo wave equation with the n-term time-fractional derivatives whose orders belong to the intervals $(1, 2]$ or $[2, 3)$, respectively. Treeby et al. [40] proposed modeling power-law absorption and dispersion for acoustic propagation using the fractional Laplacian.

However, many practical problems also involve space-fractional derivatives, in particular, Riesz fractional derivatives [37, 38]. The main purpose of this paper is to derive the analytical solutions of the multi-term time–space Caputo–Riesz fractional advection–diffusion equations (MT-TSCR-FADE) with nonhomogeneous Dirichlet boundary conditions. The multi-term time-fractional derivatives are defined in the Caputo sense, whose orders belong to the intervals $[0, 1]$, $[1, 2]$ and $[0, 2]$, respectively. These are known as multi-term time fractional diffusion terms, multi-term time fractional wave terms and multi-term time fractional mixed diffusion-wave terms.

In this paper, the derived analytical solutions are based on Luchko's Theorem [13–15]. We proposed some new techniques, such as a spectral representation of the fractional Laplacian operator and the equivalent relationship between the fractional Laplacian operator and the Riesz fractional derivative, that enabled derivation of the analytical solutions for the multi-term time–space Caputo–Riesz fractional advection–diffusion equations. As far as we are aware there are no research papers in the published literature written on this topic.

The remainder of this paper is organized as follows. In Section 2, we give some relevant definitions and lemmas. The analytical solutions of the MT-TSCR-FADE with multi-term time-fractional diffusion terms and multi-term time-fractional wave terms are derived in Sections 3 and 4, respectively. The analytical solution of the MT-TSCR-FADE with multi-term time-fractional mixed wave-diffusion terms is given in Section 5, and the conclusions of the work are presented in Section 6.

2. Background theory

For convenience, we introduce the following definitions and theorems that are used throughout this paper.

The MT-TSCR-FADE can be written in the following form:

$$P(D_t)u(x,t) = k_\beta \frac{\partial^\beta u(x,t)}{\partial |x|^\beta} + k_\gamma \frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma} + f(x,t), \tag{1}$$

in a finite domain $0 < x < L$, $t > 0$. Here x and t are the space and time variables, and k_β, k_γ are positive constants, $0 < \beta < 1$, $1 < \gamma \leqslant 2$.

The operator $P(D_t)u(x,t)$ is defined as

$$P(D_t)u(x,t) = \left(D_t^\alpha + \sum_{i=1}^{s} a_i D_t^{\alpha_i}\right) u(x,t), \tag{2}$$

$0 \leqslant \alpha_s < \cdots < \alpha_1 < \alpha \leqslant 1$ or $1 \leqslant \alpha_s < \cdots < \alpha_1 < \alpha \leqslant 2$ or $0 \leqslant \alpha_s < \cdots < \alpha_{h_0 - 1} \leqslant 1 < \alpha_{h_0} < \cdots < \alpha_1 < \alpha \leqslant 2$, and $a_i \in \mathbb{R}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, $D_t^{\alpha_i}$ is a Caputo fractional derivative of order α_i with respect to t, which is defined as follows:

$$D_t^{\alpha_i} f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha_i)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{1+\alpha_i-m}}\, d\tau, & m-1 < \alpha_i < m, \\ \frac{d^m}{dt^m} f(t), & \alpha_i = m \in \mathbb{N}. \end{cases} \tag{3}$$

The space fractional derivatives $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ and $\frac{\partial^\gamma u(x,t)}{\partial |x|^\gamma}$ are Riesz space-fractional derivatives of order β, γ, respectively, that are defined by Gorenflo and Mainardi [6] in Definition 1.

Definition 1. (See [6].) The Riesz fractional operator for $n \in \mathbb{N}$, $n-1 < \beta \leqslant n$, on a finite interval $0 \leqslant x \leqslant L$ is defined as

$$\frac{\partial^\beta}{\partial |x|^\beta} u(x,t) \equiv D^\beta u(x,t) = -c_\beta \left({}_0D_x^\beta + {}_xD_L^\beta\right) u(x,t), \tag{4}$$

where the coefficient $c_\beta = \frac{1}{2\cos(\frac{\beta\pi}{2})}$, $\beta \neq 1$, and

$${}_0D_x^\beta u(x,t) = \frac{1}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_0^x \frac{u(\xi,t)\, d\xi}{(x-\xi)^{\beta+1-n}}, \tag{5}$$

$${}_xD_L^\beta u(x,t) = \frac{(-1)^n}{\Gamma(n-\beta)} \frac{\partial^n}{\partial x^n} \int_x^L \frac{u(\xi,t)\, d\xi}{(x-\xi)^{\beta+1-n}}, \tag{6}$$

are the left-side and right-side Riemann–Liouville fractional derivatives, respectively. In particular, when $n = 2$, ${}_0D_x^\beta u(x,t) = {}_xD_L^\beta u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t)$.

If $0 \leqslant \alpha_s < \cdots < \alpha_1 < \alpha \leqslant 1$, Eq. (1) is a generalized time and space fractional advection–diffusion equation with multi-term time fractional diffusion terms. When we discuss its solutions, the initial condition is given as follows:

$$u(x,0) = \phi(x), \quad 0 < x < L. \tag{7}$$

In this case, if $a_i = 0$, $i = 1, 2, \ldots, s$, $\alpha = 1$, then Eq. (1) becomes a space fractional advection–diffusion equation with the Riesz space fractional derivatives, which was discussed by Yang et al. [43].

If $1 \leqslant \alpha_s < \cdots < \alpha_1 < \alpha \leqslant 2$, then, Eq. (1) becomes a generalized time and space fractional advection–diffusion equation with multi-term time fractional wave terms. In this case, the initial conditions are given as follows:

$$u(x,0) = \phi(x), \qquad u_t(x,0) = \psi(x), \quad 0 < x < L. \tag{8}$$

If $0 \leqslant \alpha_s < \cdots < \alpha_{h_0} \leqslant 1 < \alpha_{h_0-1} < \cdots < \alpha \leqslant 2$, then Eq. (1) becomes a generalized time and space fractional advection–diffusion equation, which we refer to as a multi-term time fractional mixed wave-diffusion equation. In this case, the initial conditions are also given by (8).

Definition 2. (See [13].) A real or complex-valued function $f(x)$, $x > 0$, is said to be in the space C_α, $\alpha \in R$, if there exists a real number p, $p > \alpha$, such that

$$f(x) = x^p f_1(x),$$

where $f_1(x)$ is in $C[0, \infty)$.

Definition 3. (See [13].) A function *f(x)*, $x > 0$, is said to be in the space C_α^m, $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, if and only if $f^m \in C_\alpha$.

Lemma 1. *For a function $u(x,t)$ defined on a finite domain $[0, L; 0, T]$, and $u(0,t) = u(L,t) = 0$, the following equality holds*:

$$-(-\triangle)^{\frac{\beta}{2}} u(x,t) = -c_\beta \big[{}_0D_x^\beta u(x,t) + {}_xD_L^\beta u(x,t)\big] = \frac{\partial^\beta}{\partial |x|^\beta} u(x,t), \tag{9}$$

where the coefficient $c_\beta = \frac{1}{2\cos(\frac{\beta\pi}{2})}$, $0 < \beta < 1$, or $1 < \beta < 2$, and the space fractional derivative $\frac{\partial^\beta u(x,t)}{\partial |x|^\beta}$ is a Riesz space-fractional derivative of order β respectively.

Proof. See [43]. □

Definition 4. (See [13].) A multivariate Mittag–Leffler (n-dimensional) function is defined as

$$E_{(a_1,\dots,a_n),b}(z_1,\dots,z_n) \equiv \sum_{k=0}^{\infty} \sum_{\substack{l_1+\dots+l_n=k \\ l_1 \geqslant 0,\dots,l_n \geqslant 0}} \frac{k!}{l_1! \cdots l_n!} \frac{\prod_{i=1}^n z_i^{l_i}}{\Gamma(b + \sum_{i=1}^n a_i l_i)}, \tag{10}$$

in which $b > 0$, $a_i > 0$, $|z_i| < \infty$, $i = 1, \dots, n$.

Lemma 2. *Let $\mu > \mu_1 > \dots > \mu_n \geqslant 0$, $m_i - 1 < \mu_i \leqslant m_i$, $m_i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, $d_i \in R$, $i = 1, \dots, n$. The initial value problem*

$$\begin{cases} \big(D^\mu y\big)(x) - \displaystyle\sum_{i=1}^n d_i \big(D^{\mu_i} y\big)(x) = g(x), \\ y^{(k)}(0) = c_k \in R, \quad k = 0, \dots, m-1,\ m-1 < \mu \leqslant m, \end{cases} \tag{11}$$

where the function $g(x)$ is assumed to lie in C_{-1}, if $\mu \in \mathbb{N}$, and in C_{-1}^1, if $\mu \notin \mathbb{N}$, and the unknown function $y(x)$ is to be determined in the space C_{-1}^m, has the solution

$$y(x) = y_g(x) + \sum_{k=0}^{m-1} c_k u_k(x), \quad x \geqslant 0, \tag{12}$$

where

$$y_g(x) = \int_0^x t^{\mu-1} E_{(.),\mu}(t) g(x-t)\, dt, \tag{13}$$

and

$$u_k(x) = \frac{x^k}{k!} + \sum_{i=l_k+1}^n d_i x^{k+\mu-\mu_i} E_{(.),k+1+\mu-\mu_i}(x), \quad k = 0, \dots, m-1, \tag{14}$$

fulfills the initial conditions $u_k^{(l)}(0) = \delta_{kl}$, $k, l = 0, \dots, m-1$. The function

$$E_{(.),\beta}(x) = E_{\mu-\mu_1,\dots,\mu-\mu_n,\beta}\big(d_1 x^{\mu-\mu_1}, \dots, d_n x^{\mu-\mu_n}\big) \tag{15}$$

is a particular case of the multivariate Mittag–Leffler function (*see* [13]) *and the natural numbers l_k, $k = 0, \dots, m-1$, are determined from the condition*

$$\begin{cases} m_{l_k} \geqslant k+1, \\ m_{l_k+1} \leqslant k. \end{cases} \tag{16}$$

In the case $m_i \leqslant k$, $i = 1, \dots, n$, we define $l_k := 0$, and if $m_i \geqslant k+1$, $i = 1, \dots, n$, then $l_k := n$.

Proof. See [13]. □

Definition 5. (See [9].) Suppose the Laplacian $(-\triangle)$ has a complete set of orthonormal eigenfunction φ_n corresponding to eigenvalues λ_n^2 on a bounded region $\mathcal{D}$, i.e., $(-\triangle)\varphi_n = \lambda_n^2\varphi_n$ on a bounded region $\mathcal{D}$; $\mathcal{B}(\varphi) = 0$ on $\partial\mathcal{D}$, where $\mathcal{B}(\varphi)$ is one of the standard three homogeneous boundary conditions. Let

$$\mathcal{F} = \left\{ f = \sum_{n=1}^{\infty} c_n\varphi_n,\, c_n = \langle f, \varphi_n\rangle,\, \sum_{n=1}^{\infty} |c_n|^2|\lambda_n|^\beta < \infty \right\}, \tag{17}$$

then for any $f \in \mathcal{F}$, $(-\triangle)^{\frac{\beta}{2}}$ is defined by

$$(-\triangle)^{\frac{\beta}{2}} f = \sum_{n=1}^{\infty} c_n(\lambda_n)^\beta \varphi_n. \tag{18}$$

Lemma 3. *(See [9].) Suppose the one-dimensional Laplacian $(-\triangle)$ defined with Dirichlet boundary condition at $x = 0$ and $x = L$ has a complete set of orthonormal eigenfunctions φ_n corresponding to eigenvalues λ_n^2 on a boundary region $\Omega = [0, L]$, if $(-\triangle)\varphi_n = \lambda_n^2\varphi_n$ then, the eigenvalues are given by $\lambda_n^2 = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, \ldots$ and the corresponding eigenfunctions are nonzero scalar multiples of $\varphi_n = \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})$.*

3. Analytical solution of the MT-TSCR-FADE with multi-term time fractional diffusion terms

In this section, we consider the analytical solution of the MT-TSCR-FADE (1) with multi-term time fractional diffusion terms. In this case, $0 \leqslant \alpha_s < \cdots < \alpha_1 < \alpha \leqslant 1$ in Eq. (1). The equation satisfies initial condition (7) and the following Dirichlet boundary conditions

$$\begin{cases} u(0,t) = \mu_1(t), & t \geqslant 0, \\ u(L,t) = \mu_2(t), & t \geqslant 0, \end{cases} \tag{19}$$

where $\mu_1(t), \mu_2(t)$ are nonzero smooth functions with order-one continuous derivatives. In order to solve the problem with nonhomogeneous boundary conditions, we firstly transform the nonhomogeneous condition into a homogeneous boundary condition. Let

$$u(x,t) = W(x,t) + V(x,t), \tag{20}$$

where

$$V(x,t) = \frac{\mu_2(t) - \mu_1(t)}{L}x + \mu_1(t), \tag{21}$$

satisfy the boundary conditions

$$\begin{cases} V(0,t) = \mu_1(t), \\ V(L,t) = \mu_2(t). \end{cases} \tag{22}$$

Using Lemma 1 and substituting (20) into (1), the function $W(x,t)$ is the solution of the following homogeneous problem:

$$\begin{cases} P(D_t)W(x,t) + k_\beta(-\triangle)^{\frac{\beta}{2}}W(x,t) + k_\gamma(-\triangle)^{\frac{\gamma}{2}}W(x,t) = f_1(x,t), & 0 < x < L,\ t > 0, \\ W(x,0) = \phi_1(x), & 0 \leqslant x \leqslant L, \\ W(0,t) = W(L,t) = 0, & t \geqslant 0, \end{cases} \tag{23}$$

with

$$\begin{cases} f_1(x,t) = -P(D_t)V(x,t) + k_\beta\dfrac{\partial^\beta}{\partial|x|^\beta}V(x,t) + k_\gamma\dfrac{\partial^\gamma}{\partial|x|^\gamma}V(x,t) + f(x,t), \\ \phi_1(x) = \phi(x) - \dfrac{\mu_2(0) - \mu_1(0)}{L}x - \mu_1(0), \end{cases} \tag{24}$$

using Definition 1 and applying the fractional derivative formulas.

We assume that the solution of the homogeneous equation in (23) has the form

$$W(x,t) = X(x)T(t). \tag{25}$$

Substituting this expression for $W(x,t)$ in (23), we obtain the Sturm–Liouville problem

$$-k_\beta(-\triangle)^{\frac{\beta}{2}}X(x) - k_\gamma(-\triangle)^{\frac{\gamma}{2}}X(x) + \lambda X(x) = 0,$$
$$X(0) = 0, \qquad X(L) = 0, \tag{26}$$

and a fractional ordinary linear differential equation with Caputo derivatives for $T(t)$, namely

$$P(D_t)T(t) + \lambda T(t) = 0, \tag{27}$$

$$T(0) = \bar{\phi}_0(x), \qquad \frac{dT(0)}{dt} = \bar{\phi}_1(x), \tag{28}$$

where the parameter λ is a positive constant.

The Sturm–Liouville problem given by (26) and (27) has eigenvalues

$$\lambda_n^2 = \frac{n^2\pi^2}{L^2}, \quad n = 1, 2, \ldots, \tag{29}$$

and corresponding eigenfunctions

$$X_n(x) = \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, \ldots. \tag{30}$$

Based on Definition 4 and Lemma 3, we set

$$W(x,t) = \sum_{n=1}^{\infty} c_{n1}(t) \sin\left(\frac{n\pi x}{L}\right). \tag{31}$$

In order to determine $c_{n1}(t)$, we expand $f_1(x,t)$ as a Fourier series using the eigenfunction $\{\sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})\}_{n=1}^{\infty}$ as

$$f_1(x,t) = \sum_{n=1}^{\infty} f_{n1}(t) \sin\left(\frac{n\pi x}{L}\right), \tag{32}$$

where

$$f_{n1}(t) = \frac{2}{L}\int_0^L f_1(x,t)\sin\left(\frac{n\pi x}{L}\right)dx. \tag{33}$$

Using Definition 3 and substituting (31) and (32) into (23), we obtain

$$P(D_t)\sum_{n=1}^{\infty} c_{n1}(t)\sin\left(\frac{n\pi x}{L}\right) + k_\beta \sum_{n=1}^{\infty} c_{n1}(t)\lambda_n^\beta \sin\left(\frac{n\pi x}{L}\right) + k_\gamma \sum_{n=1}^{\infty} c_{n1}(t)\lambda_n^\gamma \sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty} f_{n1}(t)\sin\left(\frac{n\pi x}{L}\right), \tag{34}$$

where the λ_n are given in Lemma 3.

Then, we obtain

$$P(D_t)c_{n1}(t) + k_\beta c_{n1}(t)\lambda_{n1}^\beta + k_\gamma c_{n1}(t)\lambda_{n1}^\gamma = f_{n1}(t). \tag{35}$$

Using the initial condition

$$W(x,0) = \phi_1(x), \quad 0 \leqslant x \leqslant L, \tag{36}$$

we have

$$\sum_{n=1}^{\infty} c_{n1}(0)\sin\left(\frac{n\pi x}{L}\right) = \phi_1(x), \quad 0 < x < L, \tag{37}$$

and therefore

$$c_{n1}(0) = \frac{2}{L}\int_0^L \phi_1(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots. \tag{38}$$

According to Lemma 2, we obtain

$$c_{n1}(t)=\int_0^t G_\alpha^n(\tau)\tau^{\alpha-1}f_{n1}(t-\tau)\,d\tau+c_{n1}(0)u_0(t), \tag{39}$$

where

$$G_\eta^n(t)=E_{(\nu_1,\ldots,\nu_s,\alpha),\eta}\left(-a_1t^{\nu_1},\ldots,-a_st^{\nu_s},-\kappa_nt^\alpha\right), \tag{40}$$

$$\kappa_n=k_\beta\lambda_n^\beta+k_\gamma\lambda_n^\gamma,\qquad \nu_i=\alpha-\alpha_i,\quad i=1,\ldots,s, \tag{41}$$

$$u_0(t)=1-\kappa_nt^\alpha G_{1+\alpha}^n(t). \tag{42}$$

Thus, the solution of problem (23) is given by

$$W(x,t)=\sum_{n=1}^\infty c_{n1}(t)\sin\left(\frac{n\pi x}{L}\right), \tag{43}$$

where $c_{n1}(t)$ is given in (39).

Finally, the analytical solution of the MT-TSCR-FADE (1) with multi-term time fractional diffusion terms is

$$u(x,t)=\frac{(\mu_2(t)-\mu_1(t))x}{L}+\mu_1(t)+\sum_{n=1}^\infty\left[\int_0^t G_\alpha^n(\tau)\tau^{\alpha-1}f_{n1}(t-\tau)\,d\tau+c_{n1}(0)u_0(t)\right]\sin\left(\frac{n\pi x}{L}\right). \tag{44}$$

4. Analytical solution of the MT-TSCR-FADE with multi-term time-fractional wave terms

In this section, we consider the analytical solution of the MT-TSCR-FADE with multi-term time-fractional wave terms. In this case, $1\leqslant\alpha_s<\cdots<\alpha_1<\alpha\leqslant 2$ in Eq. (1). The boundary and initial conditions are (19) and (8), respectively.

In a similar manner as presented in Section 3, in order to solve the problem with nonhomogeneous boundary conditions, we firstly transform the nonhomogeneous problem into a homogeneous problem.

Using Lemma 1 and substituting (20) into (1), the function $W(x,t)$ is the solution of the problem:

$$\begin{cases} P(D_t)W(x,t)+k_\beta(-\triangle)^{\frac{\beta}{2}}W(x,t)+k_\gamma(-\triangle)^{\frac{\gamma}{2}}W(x,t)=f_1(x,t), & 0<x<L,\ t>0,\\ W(x,0)=\phi_1(x),\qquad \dfrac{\partial W(x,0)}{\partial t}=\psi_1(x), & 0\leqslant x\leqslant L,\\ W(0,t)=W(L,t)=0, & t\geqslant 0, \end{cases} \tag{45}$$

with

$$\begin{cases} f_1(x,t)=-P(D_t)V(x,t)+k_\beta\dfrac{\partial^\beta}{\partial|x|^\beta}V(x,t)+k_\gamma\dfrac{\partial^\gamma}{\partial|x|^\gamma}V(x,t)+f(x,t),\\ \phi_1(x)=\phi(x)-\dfrac{\mu_2(0)-\mu_1(0)}{L}x-\mu_1(0),\\ \psi_1(x)=\psi(x)-\dfrac{\mu_2'(0)-\mu_1'(0)}{L}x-\mu_1'(0). \end{cases} \tag{46}$$

Similarly, based on Definition 4, we set

$$W(x,t)=\sum_{n=1}^\infty c_{n2}(t)\varphi_n(x) \tag{47}$$

to obtain

$$P(D_t)c_{n2}(t)+k_\beta c_{n2}(t)\lambda_{n1}^\beta+k_\gamma c_{n2}(t)\lambda_{n1}^\gamma=f_{n1}(t). \tag{48}$$

Using the initial conditions

$$W(x,0)=\phi_1(x),\qquad \frac{\partial W(x,0)}{\partial t}=\psi_1(x),\quad 0\leqslant x\leqslant L, \tag{49}$$

we have

$$\begin{cases} \sum_{n=1}^{\infty} c_{n2}(0)\sin\left(\frac{n\pi x}{L}\right) = \phi_1(x), \quad 0 < x < L, \\ \sum_{n=1}^{\infty} c'_{n2}(0)\sin\left(\frac{n\pi x}{L}\right) = \psi_1(x), \quad 0 < x < L, \end{cases} \tag{50}$$

and obtain

$$\begin{cases} c_{n2}(0) = \frac{2}{L}\int_0^L \phi_1(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots, \\ c'_{n2}(0) = \frac{2}{L}\int_0^L \psi_1(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n = 1, 2, \ldots. \end{cases} \tag{51}$$

For each value of n, (48) and (51) constitute a fractional initial value problem.

According to Lemma 2, we obtain

$$c_{n2}(t) = \int_0^t G_\alpha^n(\tau)\tau^{\alpha-1} f_{n1}(t-\tau)\,d\tau + c_{n2}(0)u_0(t) + c'_{n2}(0)u_1(t), \tag{52}$$

where

$$u_0(t) = 1 - \kappa_n t^\alpha G_{1+\alpha}^n(t), \tag{53}$$

$$u_1(t) = t - \kappa_n t^{1+\alpha} G_{2+\alpha}^n(t). \tag{54}$$

Thus, the solution of problem (45) is given by

$$W(x,t) = \sum_{n=1}^{\infty} c_{n2}(t)\sin\left(\frac{n\pi x}{L}\right), \tag{55}$$

where $c_{n2}(t)$ is given in (52).

Finally, the analytical solution of the MT-TSCR-FADE (1) with multi-term time fractional wave terms is

$$\begin{aligned} u(x,t) = {} & \sum_{n=1}^{\infty}\left[\int_0^t G_\alpha^n(\tau)\tau^{\alpha-1} f_{n1}(t-\tau)\,d\tau + c_{n2}(0)u_0(t) + c'_{n2}(0)u_1(t)\right]\sin\left(\frac{n\pi x}{L}\right) \\ & + \frac{(\mu_2(t)-\mu_1(t))x}{L} + \mu_1(t). \end{aligned} \tag{56}$$

5. Analytical solutions of the MT-TSCR-FADE with multi-term time-fractional mixed diffusion-wave terms

In this section, we consider the analytical solution of the MT-TSCR-FADE with multi-term time-fractional mixed diffusion-wave terms. In this case, $0 \leqslant \alpha_s < \cdots < \alpha_{h_0} \leqslant 1 < \alpha_{h_0-1} < \cdots < \alpha_1 < \alpha \leqslant 2$ in Eq. (1). The boundary and initial conditions are (19) and (8), respectively.

Similarly to the analysis in Section 4, based on Definition 4, we set

$$W(x,t) = \sum_{n=1}^{\infty} c_{n3}(t)\varphi_n(x) \tag{57}$$

to obtain

$$P(D_t)c_{n3}(t) + k_\beta c_{n3}(t)\lambda_n^\beta + k_\gamma c_{n3}(t)\lambda_n^\gamma = f_{n1}(t). \tag{58}$$

According to Lemma 2, we have

$$c_{n3}(t) = \int_0^t G_\alpha^n(\tau)\tau^{\alpha-1} f_{n1}(t-\tau)\,d\tau + c_{n3}(0)u_0(t) + c'_{n3}(0)u_1(t), \tag{59}$$

where

$$c_{n3}(0)=\frac{2}{L}\int_0^L \phi_1(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n=1,2,\ldots, \tag{60}$$

$$c'_{n3}(0)=\frac{2}{L}\int_0^L \psi_1(x)\sin\left(\frac{n\pi x}{L}\right)dx, \quad n=1,2,\ldots, \tag{61}$$

$$u_0(t)=1-\kappa_n t^{\alpha}G_{1+\alpha}^{n}(t), \tag{62}$$

$$u_1(t)=t-\sum_{i=h_0}^{s}a_i t^{1+\nu_i}G_{2+\nu_i}^{n}(t)+\kappa_n t^{1+\alpha}G_{2+\alpha}^{n}(t). \tag{63}$$

Thus, the solution of problem (45) is

$$W(x,t)=\sum_{n=1}^{\infty}c_{n3}(t)\sin\left(\frac{n\pi x}{L}\right), \tag{64}$$

where $c_{n3}(t)$ is given in (59).

Finally, the analytical solution of the MT-TSCR-FADE (1) with multi-term time-fractional diffusion-wave terms is

$$u(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t G_{\alpha}^{n}(\tau)\tau^{\alpha-1}f_{n1}(t-\tau)\,d\tau+c_{n3}(0)u_0(t)+c'_{n3}(0)u_1(t)\right\}\sin\left(\frac{n\pi x}{L}\right)$$
$$+\frac{(\mu_2(t)-\mu_1(t))x}{L}+\mu_1(t). \tag{65}$$

Special case 1. The time–space fractional telegraph equation:

$$D_t^{\alpha}u(x,t)+aD_t^{\frac{\alpha}{2}}u(x,t)=k_{\beta}(-\triangle)^{\frac{\beta}{2}}+k_{\gamma}(-\triangle)^{\frac{\gamma}{2}}+f(x,t), \tag{66}$$

with initial conditions

$$u(x,0)=\phi(x), \qquad u_t(x,0)=\psi(x), \quad 0<x<L, \tag{67}$$

and boundary conditions:

$$u(0,t)=\mu_1(t), \quad t\geqslant 0, \qquad u(L,t)=\mu_2(t), \quad t\geqslant 0, \tag{68}$$

where $1<\alpha\leqslant 2$, $0<\beta<1$ and $1<\gamma\leqslant 2$.

From (65), we obtain the solution of Eqs. (66)–(67):

$$u(x,t)=\sum_{n=1}^{\infty}\left\{\int_0^t G_{\alpha}^{n}(\tau)\tau^{\alpha-1}f_{n1}(t-\tau)\,d\tau+c_{n3}(0)u_0(t)+c'_{n3}(0)u_1(t)\right\}\sin\left(\frac{n\pi x}{L}\right)$$
$$+\frac{(\mu_2(t)-\mu_1(t))x}{L}+\mu_1(t) \tag{69}$$

where

$$u_0(t)=1+\kappa_n t^{\alpha}G_{1+\alpha}^{n}(t), \tag{70}$$

$$u_1(t)=t-at^{1+\frac{\alpha}{2}}G_{2+\frac{\alpha}{2}}^{n}(t)+\kappa_n t^{1+\alpha}G_{2+\alpha}^{n}(t), \tag{71}$$

$$G_{\alpha}^{n}(\tau)=E_{(\frac{\alpha}{2},\alpha),\alpha}\left(-a\tau^{\frac{\alpha}{2}},-\kappa_n\tau^{\alpha}\right), \tag{72}$$

$$G_{1+\alpha}^{n}(t)=E_{(\frac{\alpha}{2},\alpha),1+\alpha}\left(-at^{\frac{\alpha}{2}},-\kappa_n t^{\alpha}\right), \tag{73}$$

$$G_{2+\frac{\alpha}{2}}^{n}(t)=E_{(\frac{\alpha}{2},\alpha),2+\frac{\alpha}{2}}\left(-at^{\frac{\alpha}{2}},-\kappa_n t^{\alpha}\right), \tag{74}$$

$$G_{2+\alpha}^{n}(t)=E_{(\frac{\alpha}{2},\alpha),2+\alpha}\left(-at^{\frac{\alpha}{2}},-\kappa_n t^{\alpha}\right), \tag{75}$$

here $f_{n1}(t-\tau)$, G_{η}^{n}, κ_n, $c_{n3}(0)$, $c'_{n3}(0)$ and $E_{(\frac{\alpha}{2},\alpha),\eta}$ are as given in (33), (40), (41), (60), (61) and (10), respectively.

Special case 2. The time-fractional telegraph equation:

When $\kappa_\beta = 0$, $\kappa_\gamma = k$, $\gamma = 2$ in Eq. (66), we obtain the solution of the time-fractional telegraph equation:

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ \int_0^t E_{(\frac{\alpha}{2},\alpha),1+\alpha}\left(-a\tau^{\frac{\alpha}{2}}, -\frac{k\pi^2 n^2}{L^2}\tau^\alpha\right)\tau^{\alpha-1} f_{n1}(t-\tau)\,d\tau + c_{n3}(0)u_0(t) + c'_{n3}(0)u_1(t) \right\} \sin\left(\frac{n\pi x}{L}\right) + \frac{(\mu_2(t)-\mu_1(t))x}{L} + \mu_1(t) \tag{76}$$

where

$$\begin{aligned} u_0(t) &= 1 - \frac{k\pi^2 n^2}{L^2} t^\alpha E_{(\frac{\alpha}{2},\alpha),1+\alpha}\left(-a\tau^{\frac{\alpha}{2}}, -\frac{k\pi^2 n^2}{L^2}\tau^\alpha\right), \\ u_1(t) &= t - at^{1+\frac{\alpha}{2}} E_{(\frac{\alpha}{2},\alpha),2+\frac{\alpha}{2}}\left(-a\tau^{\frac{\alpha}{2}}, -\frac{k\pi^2 n^2}{L^2}\tau^\alpha\right) - \frac{k\pi^2 n^2}{L^2} t^{1+\alpha} E_{(\frac{\alpha}{2},\alpha),2+\alpha}\left(-a\tau^{\frac{\alpha}{2}}, -\frac{k\pi^2 n^2}{L^2}\tau^\alpha\right). \end{aligned} \tag{77}$$

This result is in accord with the result obtained in [2].

6. Conclusions

In this paper, we have proposed some new analytic techniques to solve three types of MT-TSCR-FADE with nonhomogeneous Dirichlet boundary conditions. By using Luchko's Theorem and the equivalent relationship between the Laplacian operator and the Riesz fractional derivative, we have used the spectral representation of the fractional Laplacian operator to derive analytical solutions. As far as we are aware there are no research papers in the published literature written on this topic. The methods and techniques discussed in this paper can also be applied to solve other types of multi-term fractional partial differential equations with fractional Laplacian, such as, the Szabo wave equation with the n-term time-fractional derivatives and the power-law absorption and dispersion equation for acoustic propagation with the fractional Laplacian.

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